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How many $1 \times 2 \times 4$ bricks can you get into an odd box?

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Abstract

We give a complete solution to the problem of packing $1 \times 2 \times 4$ bricks into a rectangular box of any size, the edges of the bricks being oriented parallel to those of the box. The case of major interest is when all three dimensions of the box are odd integers greater than 3.

1. Introduction

This question seems to have first been raised for cubical boxes by Foregger [5] in 1975, and a solution by Ammann and Barnes was reported by Martin Gardner in Scientific American in October 1976. Specifically, they showed that an odd n-cube could be packed with just one hole per layer if and only if $n \equiv \pm 1 \pmod{16}$. Otherwise 8 more holes (or one less brick) were necessary. In [6] Gardner adds the later comment that "Much remains unknown for non-cubical boxes" and mentions a related problem due to Klarner of whether a sufficiently high 5-square tower can be packed with one hole per layer. We shall see that it cannot, although a 7-square tower can be so packed as soon as its height is 13 or more.

The general problem of how much space must be wasted when a box with integral edges is to be packed with integral edged bricks (oriented parallel to the edges of the box) probably does not have a simple precise answer. An asymptotic solution is given by Barnes [2], where it is shown that wasted volume is bounded by a multiple of the dth power of the longest edge, where d is an integer called the variety dimension, and this value of d is best possible. In particular it is absolutely bounded if and only if $d = 0$. The precise value of the wasted area in 2 dimensions has been determined for boxes sufficiently large compared to the bricks [1].

The $1 \times 2 \times 4$ brick was variety dimension 1, and is the smallest such brick for which the packing problem becomes difficult. It has been called the 'canonical brick' in

†Sadly, the author passed away on June 10, 1994.

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several recreational articles, probably because its proportions approximate to those of bricks used in masonry.

We shall give here a complete solution to the question of the title, assuming the shortest edge exceeds 3, so that bricks may be replaced in any of the six possible orientations. For completeness, the anomalous case of height 3 boxes is dealt with in Appendix A, and the much easier problem of even-volume boxes is solved in Appendix B. Curiously, boxes of height 1, which really form a 2-dimensional problem are not exceptions to the general rule.

For an $a\times b\times c$ box with a, b, c odd we shall usually assume that $a=\min(a,b,c)$ and call it the *order* of the box. In Lemma 2.1 we introduce some simple parity conditions which prove that at least $b+c-a$ holes are necessary even for a packing with $1\times 2\times 2$ bricks. This motivates the definition of the number $\varepsilon=\varepsilon(a,b,c)$, which we call the *deficiency* of the box, as the number of bricks by which we fall short of the theoretical maximum predicted by Lemma 2.1. Specifically, ε is defined by the fact that the number of unit holes in the optimal packing of an $a\times b\times c$ box with $1\times 2\times 4$ bricks is $b+c-a+8\varepsilon$. We shall then prove the following theorem.

Theorem 1.1. *Let $a\leqslant b\leqslant c$ be odd positive integers with $a\neq 3$, and set $b=a+2r$, $c=a+2s$. Then*

(1) $\varepsilon(a,b,c)=\frac{1}{2}$ *if rs is odd*

(2) $\varepsilon(a,b,c)=1$ *if either $a\equiv 3,5,11,13$ (mod 16) and $r=0$ or $a\equiv 7,9$ (mod 16) and $(r,s)=(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$ or $(2,2)$.*

(3) $\varepsilon(a,b,c)=0$ *in all other cases.*

To show that the function given in Theorem 1.1 is an upper bound for $\varepsilon(a,b,c)$ we shall simply construct packings which achieve the desired number of holes. That it is a lower bound will follow immediately from Lemma 2.1 in those cases where $\varepsilon=0$ or $\frac{1}{2}$. The cases with $\varepsilon=1$ will require deeper analysis.

2. Preliminaries

The translate $(x,y,z)+C$ where $(x,y,z)\in\mathbb{Z}^3$ and $C=\{(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3\,|\,0\leqslant\alpha_i<1\}$ is the unit cube, will be called a *cell* in Euclidean space, and given co-ordinates (x,y,z). An $a\times b\times c$ box is then the union of the cells (x,y,z) with $0\leqslant x\leqslant a-1$, $0\leqslant y\leqslant b-1$, $0\leqslant z\leqslant c-1$. We divide the cells of Euclidean space into 64 classes according to the residue class of (x,y,z) (mod 4). These classes correspond in the obvious way to the 64 cells of the $4\times 4\times 4$ torus depicted in Fig. 1. The letters A, B, C, D refer to a grouping of the 64 classes into 4 types of 16 classes each as follows:

$A=\{(x,y,z)\,|\,x\equiv y\equiv z\ (\mathrm{mod}\ 2)\}$,
$B=\{(x,y,z)\,|\,x\equiv y\equiv z+1\ (\mathrm{mod}\ 2)\}$,
$C=\{(x,y,z)\,|\,x\equiv y+1\equiv z\ (\mathrm{mod}\ 2)\}$,
$D=\{(x,y,z)\,|\,x+1\equiv y\equiv z\ (\mathrm{mod}\ 2)\}$.

$z = 0$

3	C	B	C	B
2	A	D	A	D
1	C	B	C	B
0	A	D	A	D
	0	1	2	3

$z = 1$

3	D	A	D	A
2	B	C	B	C
1	D	A	D	A
0	B	C	B	C
	0	1	2	3

$z = 2$

3	C	B	C	B
2	A	D	A	D
1	C	B	C	B
0	A	D	A	D
	0	1	2	3

$z = 3$

3	D	A	D	A
2	B	C	B	C
1	D	A	D	A
0	B	C	B	C
	0	1	2	3

Fig. 1. The $4 \times 4 \times 4$ torus.

An empty $a \times b \times c$ box has abc holes, of which
$\frac{1}{4}(abc+a+b+c)$ are of type A,
$\frac{1}{4}(abc-a-b+c)$ are of type B,
$\frac{1}{4}(abc-a+b-c)$ are of type C,
$\frac{1}{4}(abc+a-b-c)$ are of type D.
Thus the least numerous cells are those of type D. Now it is easy to see that a $1 \times 2 \times 2$ brick in any orientation will cover exactly one cell of each type. Hence we have the following lemma.

Lemma 2.1. *An $a \times b \times c$ box with $a \leqslant b \leqslant c$, abc odd can accommodate at most $\frac{1}{4}(abc+a-b-c)$ $1 \times 2 \times 2$ bricks.*

If this number can be accommodated (and we shall see in Section 3 that it can) there will clearly be $b+c-a=a+2r+2s$ holes, where r, s are as in Theorem 1.1. Of these there will be

$a+r+s$ holes of type A,

s holes of type B, (2.1)

r holes of type C.

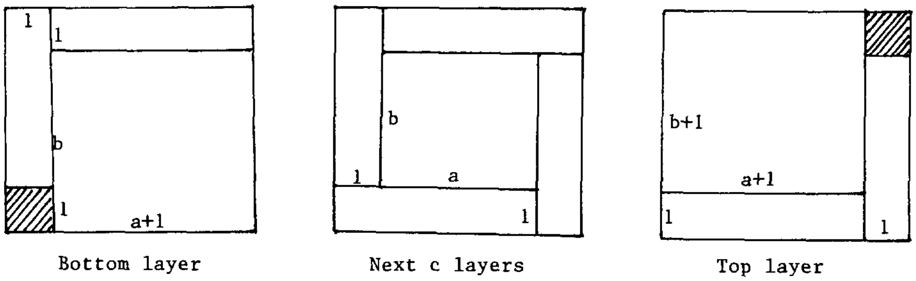


Fig. 2. Plastering an $x \times b \times c$ box.

Now $\frac{1}{4}(abc+a-b-c)=\frac{1}{4}(a^2-1)(a+2r+2s)+ars$ and since $a^2 \equiv 1 \pmod 8$ for all odd a, this number has the same parity as rs. Stealing some terminology from [3], we shall call an $a \times b \times c$ box *evil* or *odious* according as rs is even or odd, i.e. according as it will hold an even or odd number of $1 \times 2 \times 2$ bricks. Thus, the odious boxes are precisely those with $b \equiv c \equiv a+2 \pmod 4$. Since a $1 \times 2 \times 4$ brick is composed of two $1 \times 2 \times 2$ bricks, it is now clear that a packing of an evil box with $1 \times 2 \times 4$ bricks must leave at least $b+c-a$ holes, and a packing of an odious box must leave at least $b+c-a+4$ holes. We have now proved that

$$\begin{aligned} &\varepsilon(a,b,c) \geqslant 0 \quad \text{for evil boxes,} \\ &\varepsilon(a,b,c) \geqslant 1/2 \quad \text{for odious boxes.} \end{aligned} \tag{2.2}$$

3. The rules

In this section we describe several ways of converting smaller boxes into larger ones. The first of these, called *plastering* consists of attaching a 1-unit thick layer of 'plaster' around a box, to increase each edge by 2. Specifically Fig. 2 shows that an $(a+2) \times (b+2) \times (c+2)$ box may be partitioned into:

(1) An $a \times b \times c$ box concentric with it,
(2) Two $(a+1) \times (b+1) \times 1$ slabs,
(3) Two $(a+1) \times 1 \times (c+1)$ slabs,
(4) Two $1 \times (b+1) \times (c+1)$ slabs,
(5) Two unit holes in opposite corners.

Provided the slabs can be filled completely, an $a \times b \times c$ box with k holes is converted into an $(a+2) \times (b+2) \times (c+2)$ box with $k+2$ holes. We can of course fill the slabs with $1 \times 2 \times 2$ bricks since $a+1$, $b+1$, $c+1$ are all even. Hence by repeatedly plastering $1 \times 1 \times 1$ cube we see that an odd $a \times a \times a$ cube can be packed with $1 \times 2 \times 2$ bricks leaving a holes (one per layer). We can in fact arrange for these holes to run up a main diagonal of the cube. With $1 \times 2 \times 4$ bricks, we can only fill the slabs completely if at least two of $a+1$, $b+1$, $c+1$ are divisible by 4. In this case, plastering an $a \times b \times c$

box with $b+c-a+8\varepsilon(a,b,c)$ holes gives an $(a+2)\times(b+2)\times(c+2)$ box with $(b+2)+(c+2)-(a+2)+8\varepsilon(a,b,c)$ holes. We have therefore proved the following rule.

Plastering Rule. $\varepsilon(a+2,\ b+2,\ c+2)\leqslant\varepsilon(a,b,c)$ *if at least two of* a,b,c *are* $\equiv 3 \pmod 4$.

Unfortunately this rule can only be used once since the plastered box has two edges $\equiv 1 \pmod 4$, preventing a second application. Note that plastering preserves the evil/odious character of a box.

3.1. Extension

We can extend either of the two longer edges (b or c) by an even amount $2t$ by adjoining an $a\times 2t\times c$ or $a\times b\times 2t$ box. Since an odd rectangle $a\times c$ or $a\times b$ can be filled with dominoes leaving one unit square hole, this extra box can be packed with $1\times 2\times 2$ bricks leaving $2t$ holes. Hence an $a\times a\times a$ cube with a holes can be extended to an $a\times b\times a$ and then to an $a\times b\times c$ box, introducing first $b-a$ and then $c-a$ extra holes. This justifies our assertion in Section 2 that an odd $a\times b\times c$ box can indeed be packed with $1\times 2\times 2$ bricks leaving $b+c-a$ holes.

With $1\times 2\times 4$ bricks the situation is a little more complex. If t is even we can orient the bricks with their length 4 edges in the t-direction and still pack the extra box leaving $2t$ holes. In particular, for $t=2$ an $a\times b\times c$ box with $b+c-a+8\varepsilon(a,b,c)$ holes becomes an $a\times(b+4)\times c$ or $a\times b\times(c+4)$ box with $b+c+4-a+8\varepsilon(a,b,c)$. Hence we have the long extension rule.

Long extension rule. $\varepsilon(a,b+4,c)\leqslant\varepsilon(a,b,c)$ *and* $\varepsilon(a,b,c+4)\leqslant\varepsilon(a,b,c)$.

When $t=1$ we must place $1\times 2\times 4$ bricks in the $a\times 2\times c$ or $a\times b\times 2$ box with their length 2 edges in the t-direction, and so need to consider packings of an odd rectangle with 1×4 tiles.

Lemma 3.1. *Let* $a\leqslant b$ *be odd integers,* $a\neq 3$. *Then an* $a\times b$ *rectangle can be packed with* 1×4 *tiles leaving* 1 *or* 3 *holes according as* $ab\equiv 1$ *or* $3 \pmod 4$.

Proof. Place tiles parallel to whichever of a,b is $\equiv 1 \pmod 4$. Then fill the remaining width-1 strip leaving 1 or 3 holes at the end. If $a\equiv b\equiv 3 \pmod 4$, pack a 7×7 square in one corner with four 3×4 rectangles around a central hole. Then fill the rest in the obvious way.

If the $a\times c$ (resp. $a\times b$) face of an $a\times b\times c$ box can be tiled with 1×4 rectangles and one hole, we get a packing of the $a\times 2\times c$ (resp. $a\times b\times 2$) box with 2 holes, and therefore we have the following rule. □

Short Extension Rule. If $a=c \pmod 4$ *(resp.* $a\equiv b) \pmod 4)$ *and* $a\neq 3$ *then* $\varepsilon(a,b+2),c)\leqslant\varepsilon(a,b,c)$ *(resp.* $\varepsilon(a,b,c+2)\leqslant\varepsilon(a,b,c)$.

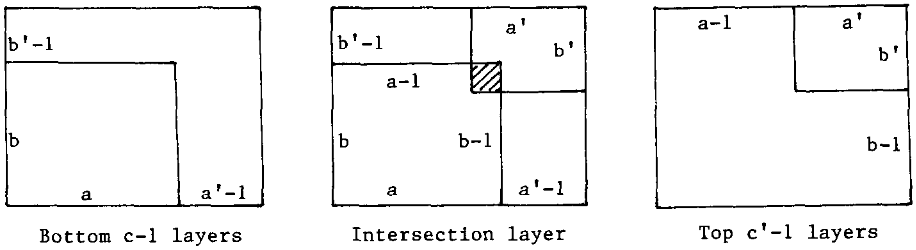


Fig. 3. Compounding an $a\times b\times c$ and $a'\times b'\times c'$ box.

It is the failure of the Short Extension Rule for $a=3$ which accounts for the anomalous behaviour of order-3 boxes.

3.2. Compounding

In this construction we place an $a\times b\times c$ box and an $a'\times b'\times c'$ box, each optimally packed with $1\times 2\times 4$ bricks in opposite corners of an $(a+a'-1)\times(b+b'-1)\times(c+c'-1)$ box. They will then overlap in a single corner cell. Provided one of the optimal packings has a corner hole, this overlap causes no difficulty. If we can fill the rest of the large box completely with $1\times 2\times 4$ bricks, we will have shown that:

$$\varepsilon(a+a'-1,b+b'-1,c+c'-1)\leqslant\varepsilon(a,b,c)+\varepsilon(a',b',c'). \tag{3.1}$$

Fig. 3 shows the compounding construction — the intervening volume to be filled with $1\times 2\times 4$ bricks consists of:

(1) An L-shape of widths $a'-1$, $b'-1$ and height $c-1$.
(2) An L-shape of widths $a-1$, $b-1$ and height $c'-1$.
(3) Two height-1 rectangular slabs $(a-1)\times(b'-1)$ and $(a'-1)\times(b-1)$.

The slabs can be filled if at least one edge is divisible by 4, and the L shapes can be filled if either the height or *both* widths are multiples of 4. This proves the following rule:

Compounding Rule. The inequality (3.1) *holds in the following cases*:

(1) $a\equiv b\equiv c\equiv 1 \pmod 4$.
(2) $a'\equiv b'\equiv c'\equiv 1 \pmod 4$.
(3) $a\equiv a'\equiv b\equiv b'\equiv 1 \pmod 4$.
(4) $a\equiv a'\equiv c\equiv c'\equiv 1 \pmod 4$.
(5) $b\equiv b'\equiv c\equiv c'\equiv 1 \pmod 4$.

Provided one of $a\times b\times c$, $a'\times b'\times c'$ has an optimal packing with a corner hole.

In the next section we shall use these rules to show that $\varepsilon(a,b,c)$ does not exceed the values given in Theorem 1.1.

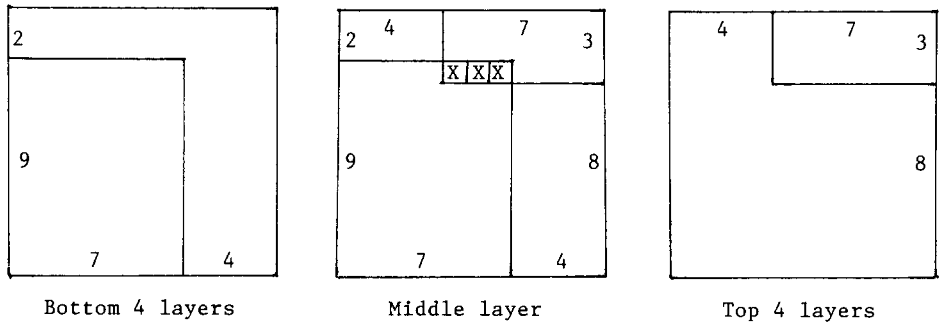


Fig. 4. $11 \times 11 \times 9$ as a generalised compound of $7 \times 9 \times 5$ and $7 \times 3 \times 5$.

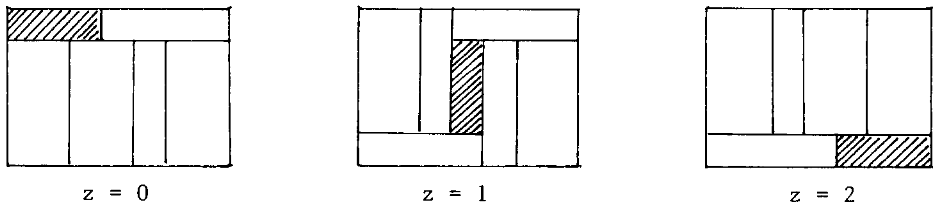


Fig. 5. A packing of $7 \times 5 \times 3$ with bricks and 9 holes.

4. Explicit constructions

We begin by establishing some particular values of $\varepsilon(a, b, c)$.

Lemma 4.1. (1) $\varepsilon(a, a, a) = 0$ *for* $a = 1, 15, 17$.

(2) $\varepsilon(a, a, a+6) = 0$ *for* $a = 7, 9$.

(3) $\varepsilon(a, a+2, a+4) = 0$ *for* $a = 3, 5, 11, 13$.

(4) $\varepsilon(a, a+2, a+2) = \frac{1}{2}$ *for* $a = 1, 3, 5, 7, 9, 11, 13, 15$.

(5) $\varepsilon(a, a, a) \leqslant 1$ *for* $a = 5, 7, 9, 11, 13, 19$.

Proof. (1) The case $a = 1$ is trivial. Case $a = 15$ follows from Fig. 8, and $a = 17$ by the Plastering Rule.

(2) Case $a = 7$ follows from Fig. 6, and $a = 9$ by the Plastering Rule.

(3) Cases $a = 3$ and $a = 11$ follow from Figs. 5 and 7 respectively, and $a = 5$ and $a = 13$ by the Plastering Rule.

(4) Only cases $a = 1, 5, 9, 13$ require proof, since the others follow from the Plastering Rule. Case $a = 1$ is trivial. Since an odd rectangle can be packed with dominoes leaving one hole, a $4 \times 5 \times 7$ or $4 \times 13 \times 15$ box can be packed with $4 \times 1 \times 2$ bricks leaving four holes. Appending these to the $3 \times 5 \times 7$ and $11 \times 13 \times 15$ with their 9 and 17 holes respectively gives packings of $7 \times 5 \times 7$ and $15 \times 13 \times 15$ with 13 and 21 holes respectively. Hence $\varepsilon(5, 7, 7) = \varepsilon(13, 15, 15) = \frac{1}{2}$, thus settling cases $a = 5$ and $a = 13$.

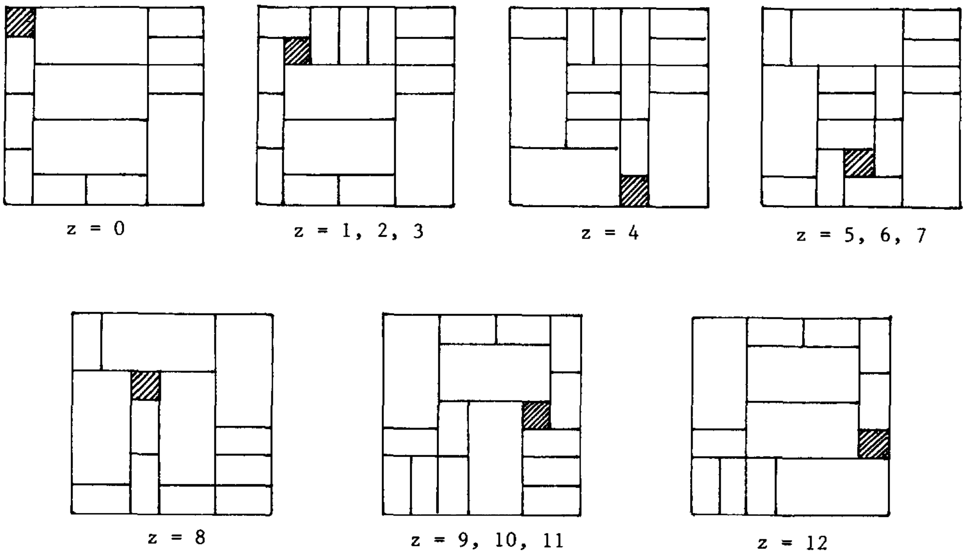


Fig. 6. A packing of $7\times7\times13$ with 78 bricks and 13 holes.

For $a=9$ refer to Fig. 4 which shows an $11\times11\times9$ box containing a $7\times9\times5$ and $7\times3\times5$ box, overlapping in 3 cells marked X. Since these are holes in the optimal packing of the $7\times3\times5$, the overlap causes no problem. Now the $7\times9\times5$ has 11 holes in its optimal packing (since $\varepsilon(5,7,9)=0$) and the $7\times3\times5$ has 9 holes, of which 3 are lost in the overlap. Since the remaining volume can clearly be packed completely with $1\times2\times4$ bricks, we have a packing of $11\times11\times9$ with $11+9-3=17$ holes, so that $\varepsilon(9,11,11)=\frac{1}{2}$.

(5) Replacing a by $a-2$ in (4) we have that $\varepsilon(a-2,a,a)=\frac{1}{2}$ for $3\leqslant a\leqslant17$, i.e. that an $(a-2)\times a\times a$ box can be packed leaving $a+6$ holes. Provided $a\neq3$, an $a\times a$ square can be packed with 1×4 tiles leaving one hole, so a $2\times a\times a$ box can be packed leaving 2 holes. Appending this to the $(a-2)\times a\times a$ box gives a packing of the a-cube with $a+8$ holes for $5\leqslant a\leqslant17$, so that $\varepsilon(a,a,a)\leqslant1$ in these cases. For $a=15,17$ we already know $\varepsilon(a,a,a)=0$. For $a=19$, apply the Compounding Rule to $5\times5\times5$ and $15\times15\times15$. □

The $17\times17\times17$, which was obtained by plastering $15\times15\times15$, has a corner hole. Hence we can apply the Compounding Rule to $a\times b\times c$ and $17\times17\times17$ to deduce that $\varepsilon(a+16), b+16, c+16)\leqslant\varepsilon(a,b,c)$, i.e. we can add 16 to all three edges of a box without increasing its deficiency. Repeated application of this principle gives the following improved version of Lemma 4.1:

Lemma 4.2. (1) $\varepsilon(a,a,a)=0$ *for all* $a\equiv\pm1 \pmod{16}$.

(2) $\varepsilon(a,a,a+6)=0$ *for all* $a\equiv7,9 \pmod{16}$.

(3) $\varepsilon(a,a+2,a+4)=0$ *for all* $a\equiv3,5,11,13 \pmod{16}$.

(4) $\varepsilon(a,a+2,a+2)=\frac{1}{2}$ *for all* a.

(5) $\varepsilon(a,a,a)\leqslant1$ *for all* a *except* $a=3$.

Now we can make use of the Extension Rules. Every odious box has $(a,b,c) \equiv (a, a+2, a+2) (\bmod 4)$, so by Lemma 4.2(4) and the Long Extension Rule, we have

All odious boxes have deficiency $\frac{1}{2}$.

Note that $a=3$ is not an exception here.

For evil boxes the situation is more complicated. Except when $a=3$, every evil box of order a can be obtained from $a \times a \times a$ by first applying the Short Extension Rule if

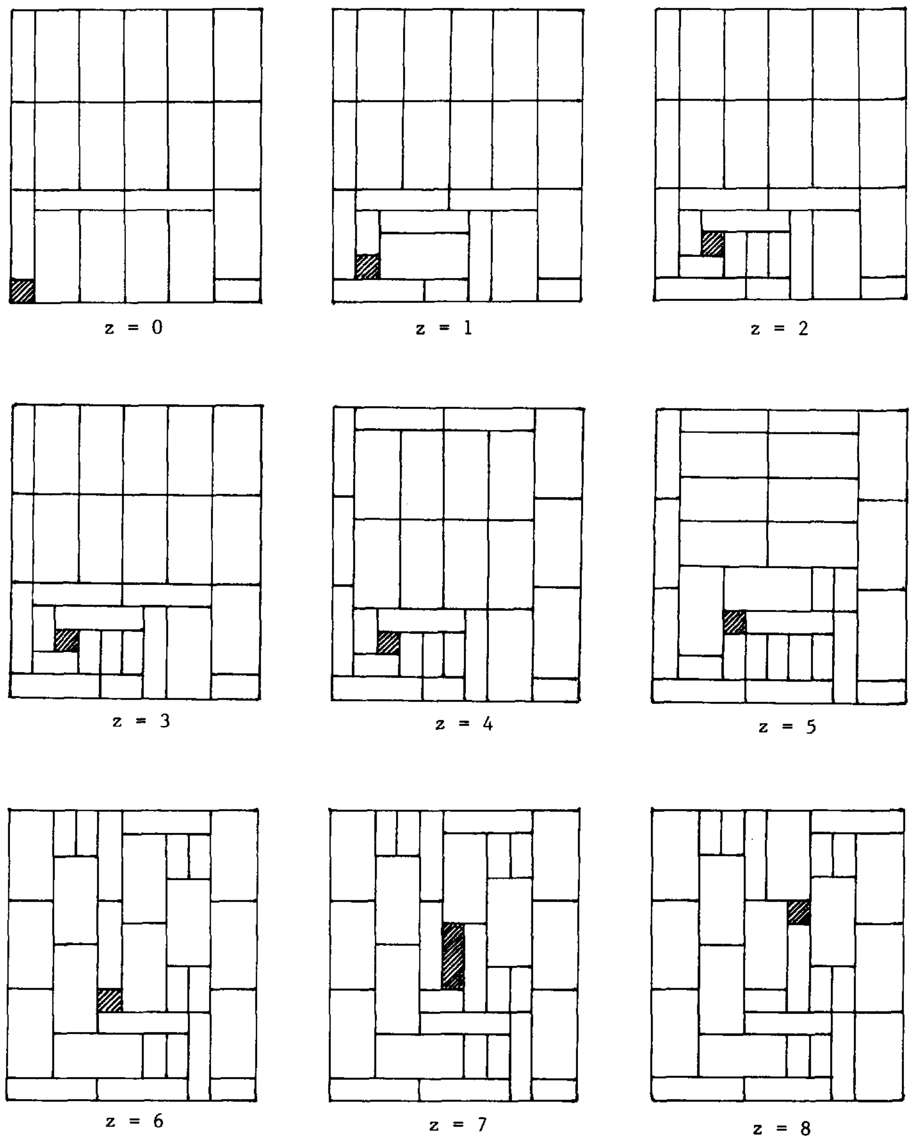


Fig. 7. A packing of $11 \times 13 \times 15$ with 266 bricks and 17 holes.

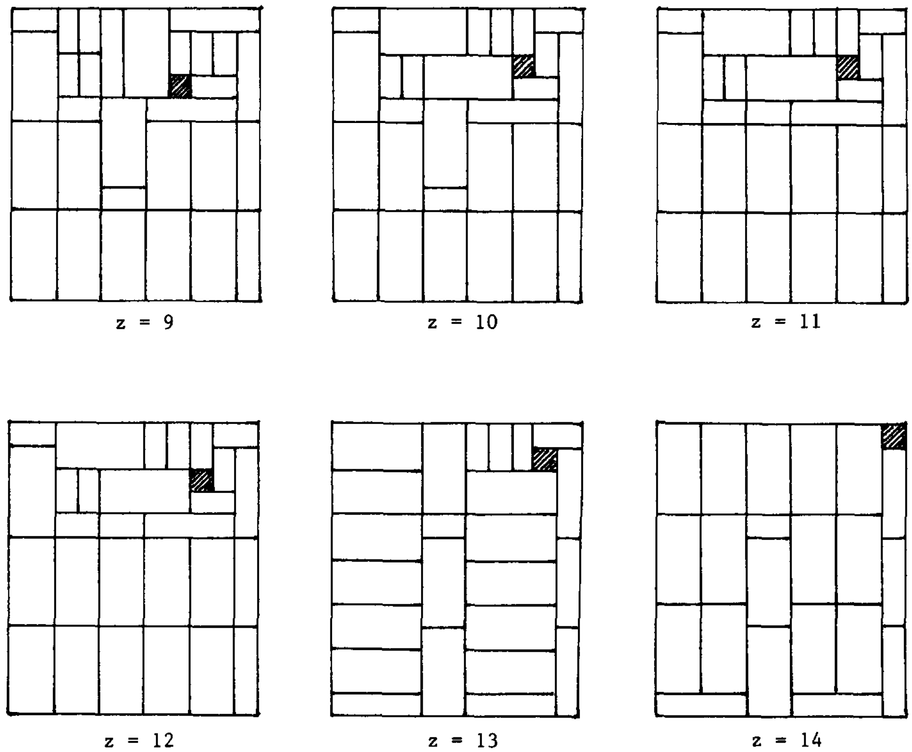


Fig. 7. Continued.

necessary, to get to $a \times a \times (a+2)$, and then using the Long Extension Rule. Hence from Lemma 4.2(1) and (5) we have

All evil boxes of order $\neq 3$ *have deficiency* 0 *or* 1.

All evil boxes of order $a \equiv \pm 1 \pmod{16}$ *have deficiency* 0.

The only evil boxes not obtainable from $a \times (a+2) \times (a+4)$ by the Extension Rules are that $a \times a \times c$ square $(c \geqslant a)$. Hence Lemma 4.2(3) implies:

If $a \equiv 3, 5, 11, 13 \pmod{16}$ *and* $a \neq 3$, *then all evil boxes of order* a, *except possibly for the square towers, have deficiency* 0.

Finally, all evil boxes $a \times b \times c$ with $c - a \geqslant 6$ are obtainable from $a \times a \times (a+6)$ by the Extension Rules, and so

All evil boxes of order $\equiv 7, 9 \pmod{16}$ *have deficiency* 0, *except possibly for the following*: $a \times a \times a$, $a \times a \times (a+2)$, $a \times a \times (a+4)$, $a \times (a+2) \times (a+4)$, $a \times (a+4) \times (a+4)$.

We have now almost proved Theorem 1.1. The only uncertain cases are the square towers of order $\equiv 3, 5 \pmod 8$, and the boxes $a \times (a+u) \times (a+v)$ where $a \equiv 7, 9 \pmod{16}$

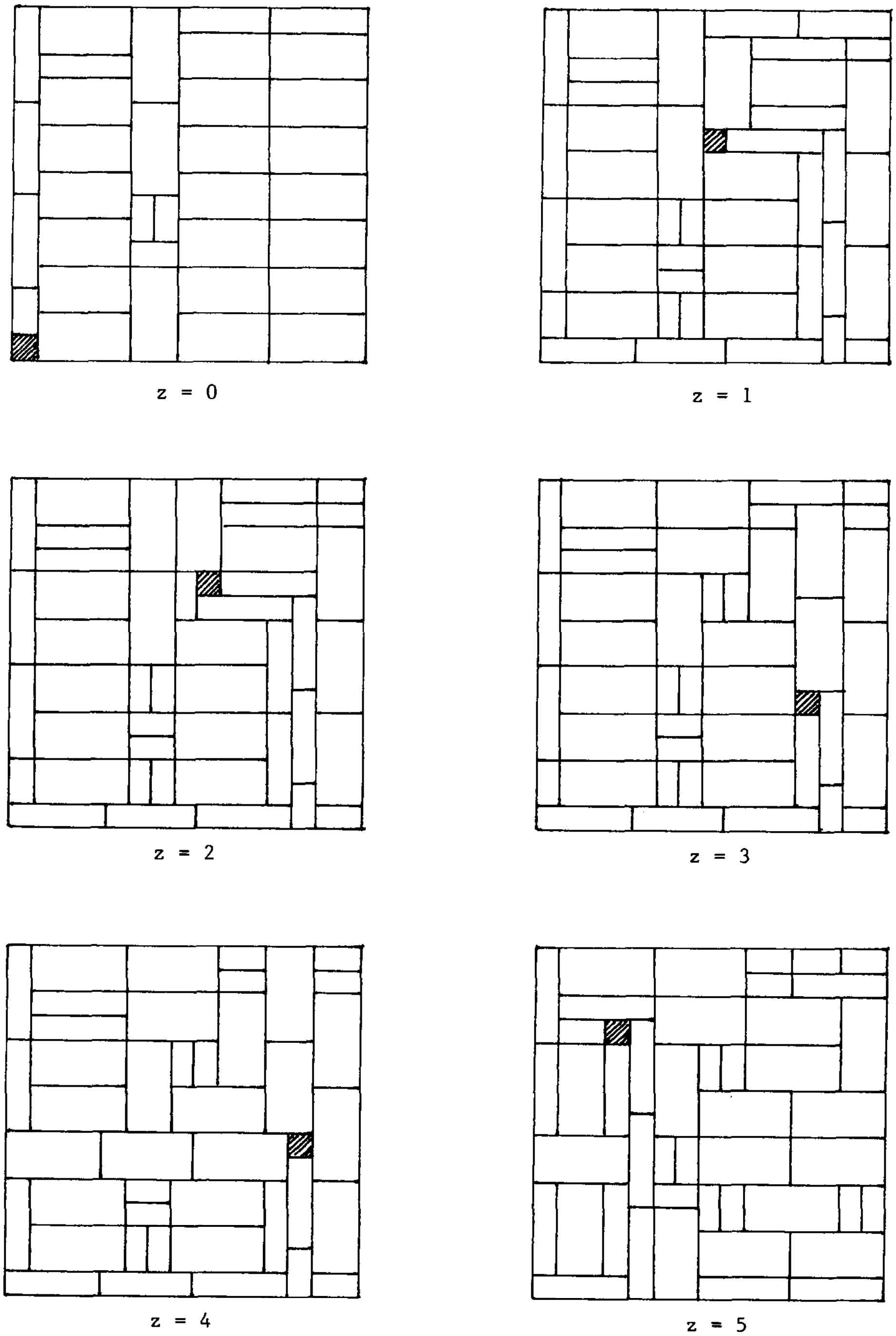
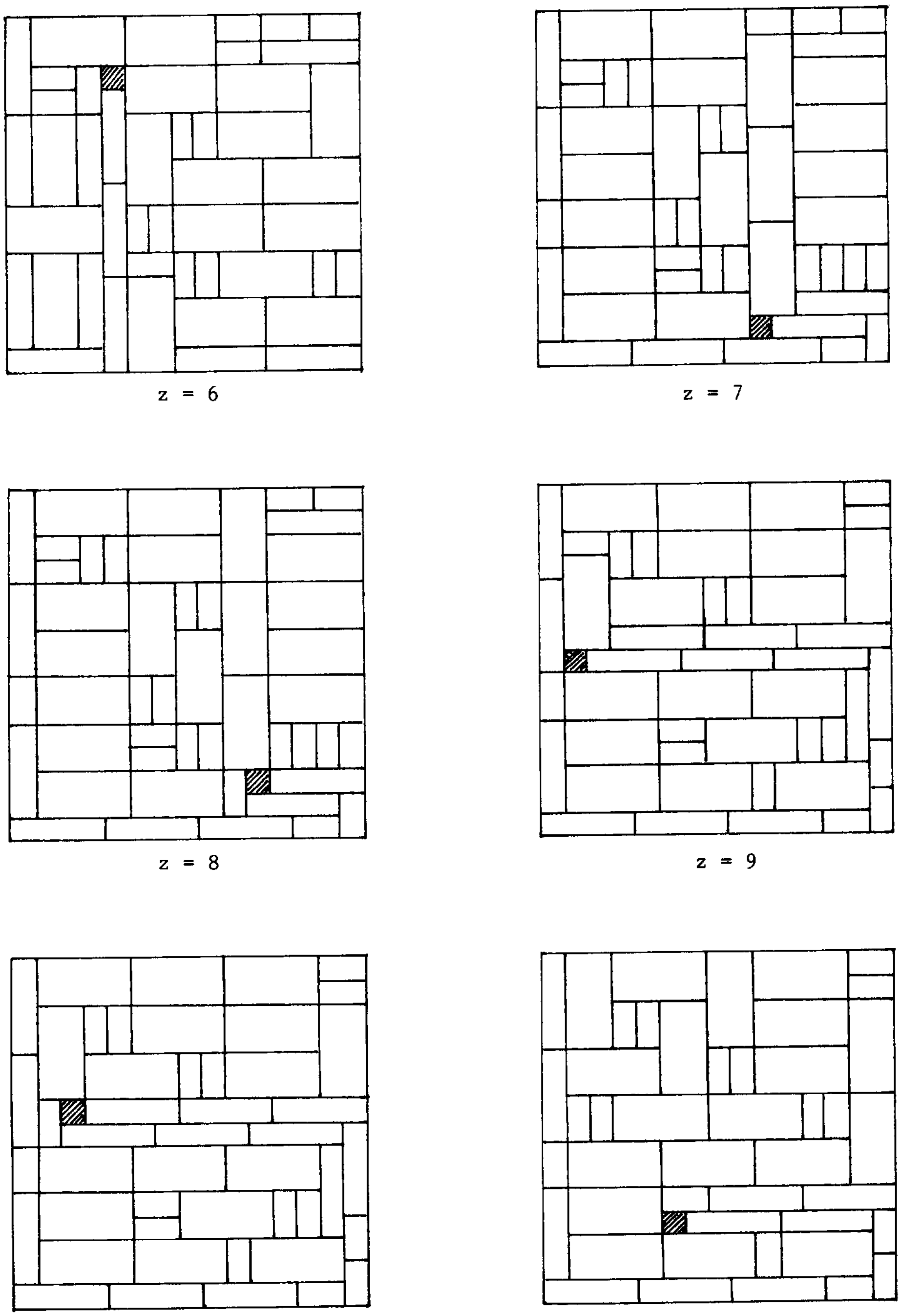


Fig. 8. A packing of $15 \times 15 \times 15$ with 420 bricks and 15 holes, discovered by Robert Ammann in 1976.



z = 6

z = 7

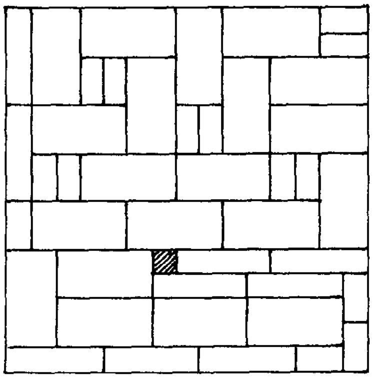
z = 8

z = 9

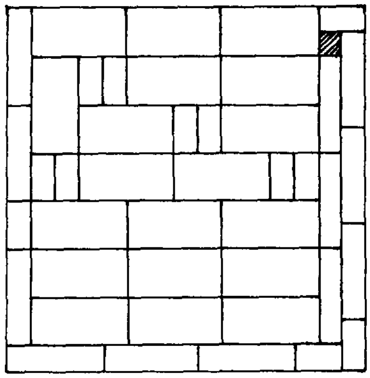
z = 10

z = 11

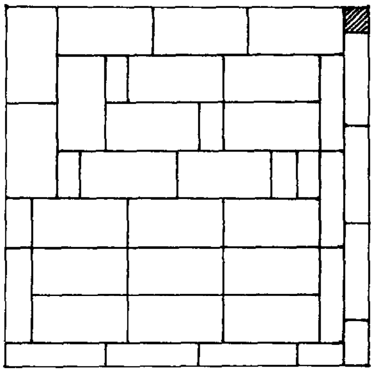
Fig. 8. Continued.



z = 12



z = 13



z = 14

Fig. 8. Continued.

and $(u,v)=(0,0),(0,2),(0,4),(2,4),(4,4)$, for which we do not yet know if the deficiency is 0 or 1. In fact all these boxes have deficiency 1, but the proof of this requires methods somewhat deeper than those used so far.

5. Torus labellings

When a box is packed with bricks, leaving holes, we can partition the holes into 64 classes according to the residue class of their co-ordinates (mod 4). In each cell of the $4\times4\times4$ torus of Fig. 1 we can place a non-negative integer indicating the number of holes of that class. Hence each packing gives rise to a labelling of the torus. We will show that the numbers in this labelling satisfy certain equations which will prove the impossibility of zero-deficiency packings of the boxes mentioned above. The methods are essentially an application of the techniques developed in [2], but the treatment given here will be self-contained. A packing with $1 \times 2 \times 4$ bricks is simultaneously a packing with $1 \times 1 \times 4$ and $1 \times 2 \times 2$ bricks, and we shall make use of the necessary conditions for each. It can be shown that the converse is true, in the sense that the simultaneous packability of a region of space (i.e. box minus holes) with $1 \times 1 \times 4$ and $1 \times 2 \times 2$ bricks implies its packability with positive and negative copies of $1 \times 2 \times 4$ bricks, but we shall not need this.

With the cell (α, β, γ) of Euclidean space we associate the monomial $x^\alpha y^\beta z^\gamma$, and with each brick or box we associate the sum of the monomials of all its cells. Thus an $a \times b \times c$ box has polynomial

$$\begin{aligned} f(x,y,z) &= \sum_{\substack{0 \leqslant \alpha < a \\ 0 \leqslant \beta < b \\ 0 \leqslant \gamma < c}} x^\alpha y^\beta z^\gamma = (1+x+x^2+\cdots+x^{a-1}) \\ &\quad \times (1+y+y^2+\cdots+y^{b-1})(1+z+z^2+\cdots+z^{c-1}) \\ &= \frac{(x^a-1)(y^b-1)(z^c-1)}{(x-1)(y-1)(z-1)}. \end{aligned}$$

The polynomial of a $1 \times 2 \times 2$ (or $2 \times 1 \times 2$ or $2 \times 2 \times 1$) brick is $(1+x)(1+y)$ or $(1+x)(1+z)$ or $(1+y)(1+z)$ according to its orientation. These are multiplied by $x^\alpha y^\beta z^\gamma$ when the brick is translated by (α, β, γ). Hence for any packing with $1 \times 2 \times 2$ bricks we have:

$$\begin{aligned} f(x,y,z) &= (1+x)(1+y)g_1(x,y,z) + (1+x)(1+z)g_2(x,y,z) \\ &\quad + (1+y)(1+z)g_3(x,y,z) + \sum_{\text{holes}} x^\alpha y^\beta z^\gamma, \end{aligned} \tag{5.1}$$

where g_1, g_2, g_3 are polynomials with integer coefficients. Setting two of the variables equal to -1 gives:

$$\left.\begin{aligned} 1+x+x^2+\cdots+x^{a-1} &= \sum_{\text{holes}} (-1)^{\beta+\gamma} x^\alpha \\ 1+y+y^2+\cdots+y^{b-1} &= \sum_{\text{holes}} (-1)^{\alpha+\gamma} y^\beta \\ 1+z+z^2+\cdots+z^{c-1} &= \sum_{\text{holes}} (-1)^{\alpha+\beta} z^\gamma \end{aligned}\right\} \text{ identically in } x, y, z. \tag{5.2}$$

The first of these expresses the fact that in each layer of the box in the x-direction, the number of holes with $\beta+\gamma$ even exceeds by 1 the number with $\beta+\gamma$ odd. Consequently in layer $x=i$ of the labelled torus we have:

$$\begin{aligned} &(\text{No. of holes with } \beta+\gamma \text{ even}) - (\text{No. of holes with } \beta+\gamma \text{ odd}) \\ &\quad = \text{Number of integers} \equiv i \pmod 4 \text{ in } \{0,1,2,\ldots a-1\}. \end{aligned} \tag{5.3}$$

Since $\beta+\gamma$ is even for holes of type A, D and odd for type B, C, this is equivalent to

$$A-B-C+D = \left\lceil \frac{a+3-i}{4} \right\rceil \quad \text{for } i=0,1,2,3, \tag{5.4}$$

where A, B, C, D denote the sum of the numbers in cells marked A, B, C, D respectively in Fig. 1 in layer $x=i$ of the torus. We refer to (5.4) as the *x-equations*. Similar analysis in the other two directions yield the *y-equations*

$$A-B+C-D=\left[\frac{b+3-i}{4}\right] \quad \text{in layer } y=i \quad \text{for } i=0,1,2,3 \tag{5.5}$$

and the *z-equations*

$$A+B-C-D=\left[\frac{c+3-i}{4}\right] \quad \text{in layer } z=i \quad \text{for } i=0,1,2,3. \tag{5.6}$$

These hold for any packing of the box with $1\times2\times2$ bricks. For $1\times1\times4$ bricks, the same method leads to a rather messy set of equations, so we shall instead use the technique of colourings. A *colouring* is defined in [2] as the assignment of a complex number called the weight to each cell in Euclidean space in such a way that whereever a brick is placed, the sum of the weights of its cells is always zero. It follows that for any packing, the weighted sum of the holes is equal to the weighted sum of all the cells in the empty box, and is therefore independent of the packing. This will be applied to some particular colourings for $1\times1\times4$ bricks, which we now describe.

The colourings we have in mind are periodic of period 4 in each direction, and so may be defined by attaching weights to the 64 cells of the $4\times4\times4$ torus.

Restricting each of x, y, z to two values only gives a set of eight cells which, if Fig. 1 were regarded as a diagram of a $4\times4\times4$ cube, would lie at the corners of a rectangular parallel piped. We shall call such a set of eight cells a block. Let us weight the cells in the block alternately 1 and -1 thus:

First layer		Second layer	
$\boxed{-1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{-1}$
$\boxed{1}$	$\boxed{-1}$	$\boxed{-1}$	$\boxed{1}$

the other 56 cells of the torus having weight zero. It is obvious that this is a colouring for $1\times1\times4$ bricks since every brick which meets the block does so in weights 1 and -1. We shall call it a *block colouring*. Specifically, we may refer to the block colouring $\{a_1,a_2\}\times\{b_1,b_2\}\times\{c_1,c_2\}$ where $\{a_1,a_2\}$, $\{b_1,b_2\}$ $\{c_1,c_2\}$ are subsets of $\{0,1,2,3\}$, and in this notation we assume the block weighted so that the cell (a_1,b_1,c_1) has weight 1. In particular, if each of $\{a_1,a_2\}$, $\{b_1,b_2\}$, $\{c_1,c_2\}$ is one of the consecutive subsets $\{0,1\}$, $\{1,2\}$, $\{2,3\}$ the block forms a small $2\times2\times2$ cube inside the $4\times4\times4$ cube of Fig. 1. (There are 27 blocks like this and their colourings are linearly independent, all other block colourings being linear combinations of them — in fact [2] shows that the entire space of colourings for $1\times4\times4$ bricks has dimension 27.) If

1st layer		2nd layer	
$\boxed{e_1}$	$\boxed{e_2}$	$\boxed{e_5}$	$\boxed{e_6}$
$\boxed{e_3}$	$\boxed{e_4}$	$\boxed{e_7}$	$\boxed{e_8}$

are the numbers occurring in any block of cells in the torus labelling for a packing with $1\times1\times4$ bricks, their weighted sum with respect to the block colouring is $(e_2+e_3+e_5+e_8)-(e_1+e_4+e_6+e_7)$ and we have just seen that this number is independent of the packing. We shall say that the block of cells is *balanced* if this number is zero, i.e. $e_2+e_3+e_5+e_8=e_1+e_4+e_6+e_7$. The property of independence of the packing may now be stated.

Lemma 5.1. *Let an $a\times b\times c$ box be packed with $1\times1\times4$ bricks in two different ways, and compute the torus labelling for each. Then if one labelling is subtracted from the other, cell by cell, all blocks will balance.*

If we compare the torus labelling of a hypothetical packing with $1\times2\times4$ bricks with that of a known packing with $1\times1\times4$ bricks we obtain 216 *balance equations* (one for each block) between the 64 integers in the torus, although as we have noted above only 27 of them are linearly independent. These together with the x, y, and z equations will give all the information we need to complete the proof of Theorem 1.1.

6. Impossibility proofs

We are now ready to attack the problem of packing $a\times a\times c$ square towers when $a\equiv3,5 \pmod 8$. It suffices to show that $\varepsilon(a,a,c)>0$ when $a\equiv5 \pmod 8$, since the case $a\equiv3 \pmod 8$ follows from the Plastering Rule. We can also assume that $c\equiv1 \pmod 4$, since the case $c\equiv3 \pmod 4$ will follow from the Short Extension Rule. If $a\equiv5 \pmod 8$ we may set $a=4k+1$ where k is odd, and suppose that $\varepsilon(a,a,c)=0$, so that there is a packing with $1\times2\times4$ bricks leaving c holes (one per layer). In the notation of Theorem 1.1, we have $r=0$, $s=\frac{1}{2}(c-a)$ and so by (2.1) all the holes are of type A and B. Hence the torus labelling for this packing has all its C and D entries zero. Let the $z=0$ layers of the torus labelling be

	e_7		e_8
e_5		e_6	
	e_3		e_4
e_1+1		e_2	

Now pack the tower in a different way with $1\times1\times4$ bricks by standing them on end so as to fill completely layers $0,1,2,\ldots c-2$. The top layer $z=c-1$ can now be packed leaving a single hole at $(0,0,c-1)$. The torus labelling for this packing consists of a solitary 1 in cell $(0,0,0)$ and 0 in the other 63 cells. Hence by Lemma 5.1 if we subtract 1 from cell $(0,0,0)$ in the torus labelling for the hypothetical packing with $1\times 2\times 4$ bricks we shall have all blocks balanced. Let the $(0,0)$ entry in layer $z=1$ of the labelling be e_1+u. Since C and D cells have zero entries, the balancing of the $\{0,1\}\times\{0,1\}\times\{0,1\}$ block gives the $(1,1)$ entry in $z=1$ as $e_3=u$. Since the $\{1,2\}\times\{0,1\}\times\{0,1\}$ block balances, the $(2,0)$ entry must be e_2+u, etc, so that layer $z=1$ looks like

$$L+\begin{array}{|c|c|c|c|}\hline & -u & & -u \\ \hline u & & u & \\ \hline & -u & & -u \\ \hline u & & u & \\ \hline \end{array} \quad \text{where } L=\begin{array}{|c|c|c|c|}\hline & e_7 & & e_8 \\ \hline e_5 & & e_6 & \\ \hline & e_3 & & e_4 \\ \hline e_1 & & e_2 & \\ \hline \end{array}.$$

Using the same argument with the $\{x_1,x_2\}\times\{y_1,y_2\}\times\{0,2\}$ and $\{x_1,x_2\}\times\{y_1,y_2\}\times\{0,3\}$ blocks shows that layers $z=2$ and $z=3$ of the torus labelling are of the form

$$L+\begin{array}{|c|c|c|c|}\hline & -v & & -v \\ \hline v & & v & \\ \hline & -v & & -v \\ \hline v & & v & \\ \hline \end{array} \quad \text{and} \quad L+\begin{array}{|c|c|c|c|}\hline & -w & & -w \\ \hline w & & w & \\ \hline & -w & & -w \\ \hline w & & w & \\ \hline \end{array}.$$

Applying the x-equation on layer $x=0$ we have

$$(e_5+e_1+1)-(e_5+e_1+2u)+(e_5+e_1+2v)-(e_5+e_1+2w)=\left[\frac{a+3}{4}\right]=k+1,$$

so that $k=-2u+2v-2w$, contradicting our assumption that k is odd. This settles the case of the square towers.

Now we must investigate boxes of size $a\times a\times a$, $a\times a\times(a+2)$, $a\times a\times(a+4)$, $a\times(a+2)\times(a+4)$ and $a\times(a+4)\times(a+4)$ when $a\equiv 7,9 \pmod{16}$. The proof of impossibility here rests on more than just a parity violation. The non-negativity of the integers in the torus labelling must also be taken into account. (In fact, unlike the square towers, these boxes could be packed with zero deficiency if we allowed the bricks to overlap, counting each overlap as a negative hole!) It suffices to show that

$\varepsilon(a, a+4, a+4)>0$ since the other cases will then follow from the Extension Rules. We can also assume $a \equiv 9 \pmod{16}$ since the case $a \equiv 7 \pmod{16}$ will follow from the Plastering Rule.

Assume then that $a=8k+1$ where k is odd, and that $\varepsilon(a, a+4, a+4)=0$, so that the box has a packing with $1 \times 2 \times 4$ bricks and $a+8$ holes. As before, it is easy to pack the box, with $1 \times 1 \times 4$ bricks with a single hole in cell $(0,0,a+3)$. Hence if 1 is subtracted from the $(0,0,0)$ entry in the torus labelling, all blocks will balance. In the notation of Theorem 1.1 we have $r=s=2$ so there are 2 B-holes, 2 C-holes, no D-holes, and $a+4$ A-holes.

The two C-holes cannot be of the same class, for if there were a solitary 2 in one C-cell (say in layer $z=i$) and all other C-cells were 0, the block of C-cells in layers $z=i$ and $z=i+2$ would not balance. Hence two of the C-entries in the torus labelling are equal to 1, and the rest are 0. The same applies to the B-entries. The D-entries of course are all zero. Furthermore, the two positive C-entries must occur in the same z-layer, for if there were a solitary 1 in the C-cells of layer $z=i$, then the block composed of the C-cells in $z=i$ together with the D-cells in $z=i+1$ would not balance. Finally, the two positive C-entries in layer $z=i$ must be adjacent in either the x or y direction in order for the block of C-cells in layers $z=i$ and $z=i+2$ to balance.

In layers $z=i+1, i+2, i+3$, all the C and D entries are zero. Suppose that after subtracting 1 from the cell $(0,0,0)$ (which is an A-cell) the layer $z=i+1$ is

	e_7		e_8
e_5		e_6	
	e_3		e_4
e_1		e_2	

.

Balancing blocks of cells spanning layers $z=i+1, i+2$ and also $z=i+1, i+3$ shows as before that layers $z=i+2$ and $z=i+3$ must be of the form

	e_7-u		e_8-u
e_5+u		e_6+u	
	e_3-u		e_4-u
e_1+u		e_2+u	

and

	e_7-v		e_8-v
e_5+v		e_6+v	
	e_3-v		e_4-v
e_1+v		e_2+v	

respectively, for some integers u, v. Without the two positive C-entries, layer $z=i$ would like

$$\begin{array}{|c|c|c|c|} \hline & e_7-w & & e_8-w \\ \hline e_5+w & & e_6+w & \\ \hline & e_3-w & & e_4-w \\ \hline e_1+w & & e_2+w & \\ \hline \end{array}\ . \tag{6.1}$$

If the two 1's in the C-cells are in the same y-layer (i.e. the same row in (6.1)), then balancing blocks which avoid this row shows that 3 rows of layer $z=i$ must coincide with (6.1). The remaining row has 1 in each blank cell, and to make blocks balance, the other two cells in that row must be augmented by 1. In short, layer $z=i$ must consist of (6.1) with one of its rows augmented by

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline \end{array}\ .$$

This augmented row will consist of cells of type B and C (from Fig. 1).

If the two 1's in the C-cells are in the same x-layer, the same argument shows that layer $z=i$ of the torus labelling (after subtracting 1 from $(0,0,0)$) consists of (6.1) with one column augmented by

$$\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}\ ,$$

this column consisting of cells of type A and C. We now deal with each case separately. If there is an augmented row, consider the y-equations. (Any one of them — they are all the same.) For simplicity we shall select the y-equation for layer $y=1$ as this avoids having to account for the extra 1 in cell $(0,0,0)$. It makes no difference whether row $y=1$ of (6.1) is augmented or not since B and C have opposite sign in the y-equations, to the four extra 1's cancel out. In layer $z=i+1$ the cells labelled e_3, e_4 are A-cells if i is even and B-cells if i is odd. Hence the y-equation says that

$$(e_3+e_4)-(e_3+e_4-2u)+(e_3+e_4-2v)-(e_3+e_4-2w)=\pm\left[\frac{b+2}{4}\right]=\pm\left[\frac{a+6}{4}\right],$$

which reduces to $2u-2v+2w=\pm(2k+1)$. This is impossible since $2k+1$ is odd.

Now suppose that here is an augmented column in (6.1), and consider the x-equation for layer $x=1$. It does not matter if column $x=1$ is the augmented column since the four extra 1's would cancel, being of types A and C. The cells labelled e_3, e_7 in $z=i+1$ are of type A or B according as i is even or odd, so the x-equation is

$$(e_3+e_7)-(e_3+e_7-2u)+(e_3+e_7-2v)-(e_3+e_7-2w)=\pm\left\lceil\frac{a+2}{4}\right\rceil,$$

which reduces to $u-v+w=\pm k$.

Since k is odd, we must have one of v, $u+w$ odd. If v is odd, the eight A and B entries in $z=i+1$ differ in parity from those in $z=i+3$, while if $u+w$ is odd, the eight A and B entries in $z=i+2$ differ in parity from those in $z=i$, except perhaps for the augmented column, and the cell $(0,0,0)$. The exceptions involve only cells of type A, so either way it implies the existence of four odd B-entries and hence at least four B-holes in the packing. This contradiction completes the proof of Theorem 1.

Remark. Given a set of cells inside an $a\times b\times c$ box, whose torus labelling satisfies both the x, y, z equations and the block balancing equations, it is easy to shuffle them around by multiplies of 4 in each direction so as to satisfy the stronger condition (5.2). Theorem 5.1 of [2] then guarantees that the remainder of the box is packable with positive and negative copies $1\times 2\times 4$ bricks. Provided the cells are well spaced out with room to manoeuvre between them it appears to be not too difficult to do it with positive bricks only — i.e. the secret of successful packing is to first find out where the holes must be, and then try to fit the bricks around them. This is how the packings of Figs. 6 and 7 were discovered. The most difficult packing of all — that of Fig. 8 — is a minor modification of a packing discovered by Robert Ammann and communicated to Martin Gardner in 1976 in response to a problem in his Mathematical Games column.

Appendix A. Boxes of order 3

To finally complete our answer to the question of the title we shall prove the following theorem.

Theorem A1. (1) *$\varepsilon(3,3,c)=1$ or 3 according as $c\equiv 1$ or $3 \pmod 4$*

(2) *If $b,c>3$ then*

$$\varepsilon(3,b,c)=\begin{bmatrix} 0 & \text{if } bc=3 \pmod 4 \\ \frac{1}{2} & \text{if } (b,c)\equiv(1,1) \pmod 4. \\ 1 & \text{if } (b,c)\equiv(3,3) \pmod 4 \end{bmatrix}.$$

Proof. Let $c=4k+d$, $d=1$ or 3. A 3×3 square will hold 4 dominoes with one hole, so by standing bricks on end we may fill the first $4k$ layers of the $3\times 3\times c$ tower with $4k$

bricks leaving one hole per layer, i.e. $4k=c-d$ holes. This leaves $9d$ holes on top, so a total of $c+8d$ holes, from which $\varepsilon(3,3,c)\leqslant d$. To show that we cannot improve on this, chessboard colour the cells in layers $z=3,7,11,\dots,4k-1$ so that each layer has 5 white and 4 black cells. In a 3×3 square tower, bricks can only stand on end so each brick intersects just one coloured layer in a domino, and therefore contains just one black cell. As there are just $4k$ black cells, the tower can accommodate no more than $4k$ bricks.

For part (2) we already know that $\varepsilon(3,5,7)=0$ and $\varepsilon(3,5,5)=\frac{1}{2}$ so the Long Extension Rule, which is valid for $a=3$, implies that $\varepsilon(3,b,c)$ is 0 if $(b,c)\equiv(1,3)$ or $(3,1)$ (mod 4), and $\frac{1}{2}$ if $(b,c)\equiv(1,1)(\text{mod } 4)$. Hence we need only consider the case $(b,c)\equiv(3,3)(\text{mod } 4), b,c,\geqslant 7$. Now a 7×7 square may be packed with four 3×4 rectangles around a central hole, so will hold 12 1×4 tiles. Hence a $7\times 7\times 2$ box will hold 12 $1\times 2\times 4$ bricks. The 7×7 square will easily hold 4 2×4 rectangles, so lying 4 bricks flat on top of the $7\times 7\times 2$ gets 16 bricks into a $7\times 7\times 3$ box, leaving 19 holes. Thus $\varepsilon(3,7,7)\leqslant 1$ and by the Long Extension Rule, $\varepsilon(3,b,c)\leqslant 1$ when $b\equiv c\equiv 3\,(\text{mod } 4)$ and $\min(b,c)\geqslant 7$. It remains to show that $\varepsilon(3,b,c)>0$ in this case, and by the Long Extension Rule it is enough to do this when $b=c$, i.e. to show that $\varepsilon(3,c,c)>0$ when $c=4k+3$.

Lebal those cells of a $c\times c$ square in which $x+y\equiv 0\,(\text{mod } 4)$ with P or Q according as x is even or odd, to form the pattern

	Q				Q			
		P				P		
			Q				Q	
P				P				P
	Q				Q			
		P				P		
			Q				Q	
P				P				P

.

A simple count shows that there are $2(k+1)^2$ P-cells and $2k(k+1)$ Q-cells. Note that

(1) Every 1×4 rectangle contains either a P-cell or a Q-cell.

(2) Every 2×4 rectangle contains both a P-cell and a Q-cell.

In a $c\times c\times 3$ box, let us colour black the P-cells in the middle layer, and the Q-cells in the top and bottom layers. Since the ceiling is too low for bricks to stand on end, each brick must intersect one layer in a 2×4 rectangle, or else intersect the middle layer and one other in the same 1×4 rectangle. Hence each brick must contain at least one black cell, so the box can accommodate no more than $2(k+1)^2+4k(k+1)=6k^2+8k+2$ bricks, filling a volume of $48k^2+64k+16$. As the volume of the box is $3c^2=48k^2+72k+27$ there must be at least $8k+11=2c+5$ holes, and so $\varepsilon(3,c,c)\geqslant 1$ as required.

Appendix B. How many $1 \times 2 \times 4$ bricks can you get into an even box?

This is one of those unpleasant problems where the abundance of special cases makes the answer almost harder to state than it is to prove. By an even box, we mean one in which abc is even. As Lemma 2.1 is no longer relevant, the first thing we need is a suitable definition of deficiency for an even box. There are abc cells, but if only one edge (say c) is even, then each odd $a \times b$ layer meets the bricks in 1×2, 1×4 and 2×4 rectangles and so must have at least one hole — hence a volume of only $(ab-1)c$ cells is available to be filled. The maximum number of bricks we can hope to pack is therefore

$$N = N(a,b,c) = \begin{cases} [\frac{1}{8}abc] & \text{if at least two of } a,b,c \text{ are even,} \\ [\frac{1}{8}(ab-1)c] & \text{if only } c \text{ is even.} \end{cases}$$

If this number cannot be attained, we may regard the box as deficient and measure its deficiency $\varepsilon = \varepsilon(a,b,c)$ by saying that the optimal packing has $N - \varepsilon$ bricks.

Any box with $(a,b,c) \equiv (2,2,2) \pmod 4$ is deficient. This seems to have been first noticed by De Bruijn who gave the following simple proof, which he later generalised to yield his classical theorem on packings with harmonic bricks [4]. First pack the box with an odd number of $2 \times 2 \times 2$ cubes and colour them chessboard fashion. A $1 \times 2 \times 4$ brick placed anywhere will contain 4 black and 4 white cells, where as the box has an excess of 8 cells of one colour. Thus 8 holes are necessary. That 8 holes suffice is clear if we combine bricks in pairs to form $2 \times 2 \times 4$ bricks. Taking the $2 \times 2 \times 2$ cube as our unit cell, the problem becomes a scaled up version of packing an odd box with $1 \times 1 \times 2$ dominoes. Clearly we can do this with just one $2 \times 2 \times 2$ hole, and so $\varepsilon(a,b,c) = .1$

Apart from these, the only other deficient boxes are certain thin slabs and towers of order 2 or 3 which do not allow bricks to lie in all six possible orientations. We identify them all and determine their deficiencies in

Theorem B1. *Let m,n,p be arbitrary non-negative integers. Then*

(1) *The following boxes have $\varepsilon = 1$:* $(4m+2) \times (4n+2) \times (4p+2)$, $3 \times (4m+2) \times (4n+2)$, $3 \times (4m+6) \times (4n+7)$, $2 \times 2 \times (4n+3)$.

(2) *The following boxes have $\varepsilon = 2$:* $2 \times 3 \times (4n+3)$, $3 \times 3 \times (4n+2)$.

(3) *All even boxes not of the above forms have $\varepsilon = 0$.*

Proof. If a,b,c are all even, we have either $(a,b,c) = (4m+2,\ 4n+2,\ 4p+2)$ for which we have already seen that $\varepsilon = 1$, or else one of a,b,c is divisible by 4, in which case it is obvious that the box may be filled completely and $\varepsilon = 0$. The rest of the proof falls into two cases:

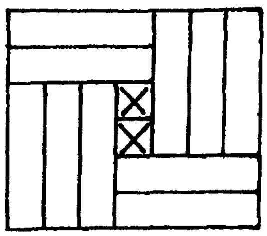
Case I: *a,b even, c odd.*

If either a or $b = 0 \pmod 4$ we can fill the box completely and $\varepsilon = 0$. Otherwise $(a,b) = (4m+2,\ 4n+2)$, $abc \equiv 4 \pmod 8$ so a zero-deficiency packing means one with just 4 holes. We can certainty pack a $(4m+2)\times(4n+2)$ rectangle with 2×4 tiles and a single 2×2 hole. Putting this on top of a completely packed $a\times b\times(c-1)$ box shows that $\varepsilon=0$ if $c\equiv 1\,(\mathrm{mod}\,4)$. Hence any deficient box of this type must have $c\equiv 3\,(\mathrm{mod}\,4)$.

Suppose first that $c=3$. Then the bricks cannot stand on end, so each brick meets each $a\times b$ layer in a 1×4 or 2×4 rectangle. We cannot pack an $a\times b$ rectangle completely with 1×4 tiles if $(a,b)\equiv(2,2)\,(\mathrm{mod}\,4)$. (This is a special case of De Bruijn's theorem — easy proof: Pack with an odd number of 2×2 squares and colour them chessboard fashion. Each 1×4 tile has 2 black and 2 white cells, but the $a\times b$ rectangle has 4 extra cells of one colour.) It follows that each of the 3 layers has at least 4 holes, so $\varepsilon\geqslant 1$. Conversely, each 2×4 rectangle is a scaled up version of a 1×2 domino, so we can in fact pack each layer leaving one 2×2 hole. Hence $\varepsilon=1$ in this case.

Next let $a=b=2$, $c\equiv 3\,(\mathrm{mod}\,4)$. In a 2×2 tower bricks can only stand on end, so each $1\times 1\times c$ column has at least 3 holes, giving 12 holes in total. Conversely it is clear that each $2\times 2\times 4$ section of the tower can be filled leaving a $2\times 2\times 3$ space on top, and therefore $\varepsilon=1$.

Any other box falling under Case I must have $\max(a,b)\geqslant 6$, $c\geqslant 7$, and a $2\times 6\times 7$ box does have a zero-deficiency packing with 4 holes thus:



2 layers

Adjoining a completely packed $2\times 6\times(c-7)$ box packs $2\times 6\times c$ leaving 4 holes, and this may be further extended to $2\times b\times c$ and then $a\times b\times c$ without introducing any more holes. Hence all boxes in Case I have $\varepsilon=0$, except for $(4m+2)\times(4n+2)\times 3$ and $2\times 2\times(4n+3)$ which have $\varepsilon=1$.

Case II: *ab odd, c even.*

An $a\times b$ rectangle may be packed with dominoes and one hole. Hence if $c\equiv 0\,(\mathrm{mod}\,4)$ standing bricks on end gives a zero deficiency packing leaving one empty $1\times 1\times c$ column. Thus we may as well assume $c\equiv 2\,(\mathrm{mod}\,4)$ and also that $a\leqslant b$.

If $ab\equiv 3\,(\mathrm{mod}\,4)$, then $(ab-1)c\equiv 4\,(\mathrm{mod}\,8)$, so a zero-deficiency packing will have $c+4$ holes. To achieve this, note that since $c-2=0\,(\mathrm{mod}\,4)$ an $a\times b\times(c-2)$ box may be packed as stated above leaving an empty $1\times 1\times(c-2)$ column of holes. Also by Lemma 3.1 an $a\times b$ rectangle is packable with 1×4 tiles and 3 holes, so $a\times b\times 2$ is packable with $1\times 4\times 2$ and 6 holes. Adjoining this to $a\times b\times(c-2)$ with $c-2$ holes gives the required packing of $a\times b\times c$ with $c+4$ holes.

If $ab\equiv 1\,(\mathrm{mod}\,4)$ and $a\neq 3$ then Lemma 3.1 again shows that an $a\times b$ rectangle is packable with 1×4 tiles and one hole, so we can pack $a\times b\times 2$ with $1\times 4\times 2$ and

2 holes. Adjoining this to $a \times b \times (c-2)$ with $c-2$ holes gives a zero-deficiency packing of $a \times b \times c$ with c holes.

We have now shown that all boxes in Case II have $\varepsilon = 0$, except possibly for those with $a = 3$, $b \equiv 3 \pmod 4$, $c \equiv 2 \pmod 4$. For these, a zero-deficiency packing if it exists, will have exactly c holes.

First let $c = 2$, so that we have a 3×2 tower, b units high. Each $3 \times 2 \times 4$ section can be filled completely, leaving a $3 \times 2 \times 3$ space of 18 holes $(= c + 16)$ on top. This is plainly the best we can do since bricks can only stand on end, and each of the six $1 \times 1 \times b$ columns must have 3 holes. Thus $\varepsilon = 2$ here.

Next let $b = 3$. Each $3 \times 3 \times 4$ section of the tower may be packed with 4 bricks on end and a column of 4 holes. Together with the empty $3 \times 3 \times 2$ space on top this gives a packing with $c-2$ bricks and $c+16$ holes. Thus $\varepsilon \leqslant 2$. To see that we cannot improve on this, chessboard colour layers $z = 3, 7, 11, \ldots, c-3$ with 4 black and 5 white cells each and use the same argument as that for 3×3 towers in Appendix A.

We now remain with the case $a = 3$, $b = 4n + 7$, $c = 4m + 6$, $m, n \geqslant 0$. For this, a packing of deficiency 1 may be constructed as follows: Four 2×4 tiles fit easily into a 7×6 rectangle, so adjoining this to the packing of $2 \times 7 \times 6$ shown above gives a packing of $3 \times 7 \times 6$ with 14 holes. Now $3 \times 7 \times (c-6)$ is packable with $c-6$ holes (since $c - 6 \equiv 0 \pmod 4$) so we can pack $3 \times 7 \times c$ leaving $c+8$ holes and hence $3 \times b \times c$ also with $c+8$ holes (because $3 \times (b-7) \times c$ can be filled completely).

It remains to prove the impossibility of a zero-deficiency packing of $3 \times (4n+7) \times (4m+6)$. For this we use the same P, Q labelling on the three $b \times c$ layers that was employed in Appendix A, colouring black the P cells in the middle layer, and the Q cells in the other two layers. Each brick contains at least one black cell, so counting ones P's and Q's imposes a limit on the number of bricks that the box will accommodate. When this limit is attained it will be found that $c+8$ cells remain empty. The details of the counting are left as an exercise.

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